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Green's Functions for the Acoustic Field in Lined Ducts with Uniform Flow

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This paper presents the analytical modeling of the radiation of sound from a moving boundary inside a lined circular duct containing uniform axial flow. This formulation can be used to model impedance discontinuities in the acoustic liners of turbofan-engine inlet ducts, such as the structural strips that separate liner panels, the liner splices near the fan, liner repair patches, and so forth. The model is based on finding a closed-form Green's function for the acoustic pressure inside the circular lined duct using a spectral expansion method, that is, using the duct modes. This method allows us to explicitly satisfy the continuity condition at the source location, which is strictly required by the Green's function. The solution is found for a point source and then extended to a finite piston by using the divergence theorem in the appropriate form to account for the effect of the soft wall. The Maxwell reciprocity principle used in this process is satisfied by using the adjoint Green's function, rather than the direct solution itself, because of the nonsymmetry introduced by the convective flow. As a consequence of the axial nonsymmetry, the expressions obtained for the radiated pressure depend on the direction of propagation of sound. The modeling of a circumferential array of rigid patches bonded to the surface of the liner is presented to illustrate the potential applications of this formulation.

Nomenclature

A	=	modal amplitude
a	=	duct radius
c	=	speed of sound
d	=	piston dimension
g	=	Green's function
I	=	acoustic intensity
i	=	imaginary unit, $i^2 = -1$
k	=	wave number, eigenvalue
L	=	linear operator, Laplacian operator for the duct cross section
L	=	linear operator, convected wave equation operator
M	=	Mach number
p	=	acoustic pressure
r	=	spatial coordinate (radial)
t	=	time
V	=	piston velocity
v	=	particle velocity
W	=	acoustic power
Z	=	transfer function/matrix
z	=	spatial coordinate (axial)
α	=	piston dimension (angular)
β	=	specific acoustic admittance
δ	=	Dirac delta
θ	=	spatial coordinate (angular)
Λ	=	modal integral factor
Φ	=	acoustic mode
ω	=	angular frequency

Subscripts

d	=	disturbance
m	=	circumferential mode order
n	=	radial mode order
o	=	observer
p	=	piston
s	=	source
w	=	wall

Superscripts

$\ell, +, -$	=	direction of propagation
$*$	=	adjoint operator, adjoint solution

I. Introduction

THE propagation of sound in lined ducts has been extensively studied over the past years, especially because of its direct application to turbofan-engine noise. In fact, the use of acoustic liners has become the most successful technique to reduce turbofan noise, and their application on aircraft engines is expected to continue. Unfortunately, the uniformity of the liner surface is often discontinued by the presence of other devices when mounted on real engine inlets. The structural strips that separate liner panels, which have recently become the focus of extensive research, are common examples of these discontinuities. Their behavior is similar to those of noise sources located at the boundary of the inlet duct producing disturbances that add to the present sound field. As a consequence, the propagation of sound inside the duct is subject to reflections and scattering of acoustic energy among both circumferential and radial propagating modes. A detailed description of this problem can be obtained by developing a model for the sound radiation inside the circular duct including the convective uniform flow and the soft-wall boundary condition. A formulation of this kind is investigated in this paper by finding a closed-form Green's function in terms of the acoustic modes of the considered circular lined duct.

Expressions for the radiation of sound inside ducts with *rigid walls* and uniform mean flow have been extensively investigated using Green's functions [1,2]. These Green's functions have been conveniently expressed as a linear combination of acoustic modes, which is also the purpose of this investigation. In addition, the solution for the case of a soft-wall duct was also found in a similar fashion but without including the convective flow [3]. However, the application of a mode series to the present scenario that includes both

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convective flow and soft-wall boundary conditions is not as straightforward as in the previous cases. The increased complexity is given by the nonorthogonal nature of the modes and the nonsymmetry introduced by the flow. As a consequence, the procedure to find the mode amplitudes that fully define the series expansion requires a more detailed analysis that will be presented in this paper.

The first studies to describe the mode propagation in lined ducts consisted of finding solutions to the equations of acoustic motion derived by Morse [4] in a rectangular cross-section duct with fluid at rest relative to the duct walls. These equations were later modified to take into account the effect of a uniform, subsonic, axial, mean flow [3] that takes place in the inlets of turbofan engines. Also, Pridmore-Brown [5] extended the analysis to include the effect of an axial fluid velocity which does vary across the duct cross section (shear flows). The present investigation is constrained to solving the propagation and radiation of sound in an infinite circular lined duct (with constant cross section) and uniform locally reacting impedance. For the case of a rectangular [6,7] and a circular [8] duct with lined walls, the appropriate boundary condition for this scenario was previously developed using continuity of pressure and particle displacement. In a more general analysis, the same boundary condition was formally derived by Myers [9] for a duct of arbitrary shape and flowfield. Although this formulation can be considered as the most general, its application to cases with constant cross sections and uniform flow parallel to the duct axis leads to expressions identical to the derivations by Ko [6,8] and Tester [7].

The solutions to the resulting boundary value problem were then investigated in terms of acoustic modes. Ducts of several cross-section shapes were investigated [6–8,10]. The characteristic equation that provides the mode eigenvalues in each case was found to be a function of the flow Mach number, the axial wave numbers, and the liner impedance, which leads one to find solutions that are complex valued and dependent upon frequency. An interesting result from this eigenvalue problem is that the hypotheses of the Sturn–Liouville theorem are not satisfied and, therefore, it is not guaranteed that the acoustic modes will form a complete orthogonal basis to describe the acoustic field inside the lined duct with flow. In fact, it is well known that the acoustic modes for this problem are not orthogonal, but the completeness of the eigenfunctions has not been strictly determined. Nevertheless, previous work by Tester [7] showed that there is enough evidence to support the completeness of the nonorthogonal series provided that convergence is always achieved as the number of modes in the expansion is increased. In particular, Tester investigated the convergence of a mode series Green's function for two-dimensional lined ducts with respect to a refined ray model evaluated in the region where the direct field from the source is dominant. The observed rate of convergence for cases with and without flow showed a strong indication that the mode series for the lined duct with flow is indeed complete. The approach to finding this Green's function was by using a Lorentz transformation that allows posing the problem as if the fluid was at rest relative to the duct walls. However, this model had limitations on the applicability of the solution for certain values of the liner impedance. More important, this method does not allow imposing the continuity of the solution at the source plane, which is a condition requirement of the acoustic field Green's function. A mode series solution for the Green's function was also investigated in three-dimensional ducts by Zorumski [11] using a method developed previously by Drischler [1] and refined to include the effect of the wall impedance. The method consisted of expressing the sound field generated by a point source as a linear combination of inverse Fourier transforms, but without forcing the continuity condition at the source plane. It must be remarked at this point that the presence of convective flow in the lined duct breaks the symmetry of the mode eigenvalues with respect to the direction of propagation and causes this continuity condition to not be naturally satisfied. Therefore, this condition must be explicitly imposed in the solution process to properly find the Green's function. The main objective of this paper is to derive an exact mode series solution for this Green's function that does satisfy the continuity at the source plane and to demonstrate its application to model liner discontinuities. The solution will be investigated for a circular duct but it can easily be extended to other cross-section geometries. The developed method will be denoted as the spectral expansion method given that all the derivations are performed by using only the duct modes and axial wave numbers without performing any Fourier transformation.

The radiated sound field from a source of finite dimensions located at the duct boundary will also be obtained using the derived Green's function for a point source. The extension of the solution to the finite region is performed using the divergence theorem that leads to integrate the Green's function over the radiating surface. Because the linear operator defining the convected wave equation with the soft-wall boundary condition is not self-adjoint, common reciprocity does not apply. For this reason, use of the adjoint Green's function will be made after applying the divergence theorem in order to have a direct formula for the sound radiation from the finite piston source. In addition, the expressions for the radiated sound field will be derived in closed form for the case of rectangular-shaped finite sources.

As mentioned before, the formulation derived in this paper can be used for a number of applications. The modeling of a particular application is developed by correctly adjusting the intensity or velocity of the source. To illustrate this process, an example consisting of a circumferential array of rigid patches bonded to the lined duct surface is described.

II. Sound Radiation Formulation

The problem consists of finding the sound field in a lined circular duct generated by the motion of a finite boundary. This motion can be interpreted as the perturbation produced by a discontinuity in the surface of acoustic liners, such as the ones that are usually present in the intakes of turbofan engines, that is, liner splices or rigid structural panels. The radiated acoustic pressure from the moving boundaries can be expressed in terms of the Green's function for the convected wave equation applied to the lined circular duct. Because there is no sound source within the medium, the solution is found as a surface integral over the radiating sectors as follows:

$$p_{\mathrm{rad}} \propto \int_{S_{\mathrm{rad}}} v_s \cdot g(\boldsymbol{r} \mid \boldsymbol{r}_s)\, \mathrm{d}S_{\mathrm{rad}} \tag{1}$$

The input velocity v_s of the moving boundary has to be determined accurately for each application. This requires a difficult analysis because the induced radial particle velocity at the liner surface does not vanish in general. The particular case of a rigid patch bonded to the liner surface will be used to describe this analysis.

The Green's function in Eq. (1), $g(\boldsymbol{r} \mid \boldsymbol{r}_s)$, was previously investigated in closed form using a wave number (Fourier) transform approach without addressing the continuity at the source plane [11]. A recent investigation using this method shows that the solution exhibits a discontinuous behavior for a finite number of expansion terms and the main contribution to this discontinuity is given by a single term (for each circumferential order) identified as a strongly decaying surface wave [12]. It appears that when these terms are included, the discontinuity is reduced significantly. Unfortunately, this surface wave can only be captured by solving a significantly large number of radial order eigenvalues, which could result in impractical common industrial applications. In addition, there is an ambiguity regarding these terms given that they have also been identified to represent possible instabilities. A brief discussion about these "strange" terms will follow later in the paper. On the other hand, the approach taken in this investigation consists of finding an exact closed-form expression for the Green's function using a spectral (modal) expansion method that explicitly allows imposing the continuity condition [13]. This approach is illustrated in Fig. 1. The first step consists of formulating the homogenous problem and solving for the lined duct propagating modes (eigenfunctions). This process involves the derivation of the soft-wall boundary condition equation as well as an efficient scheme to compute the eigenvalues.

Then, because the problem is linear, the Green's function is found as a superposition of the propagating modes inside the lined duct.

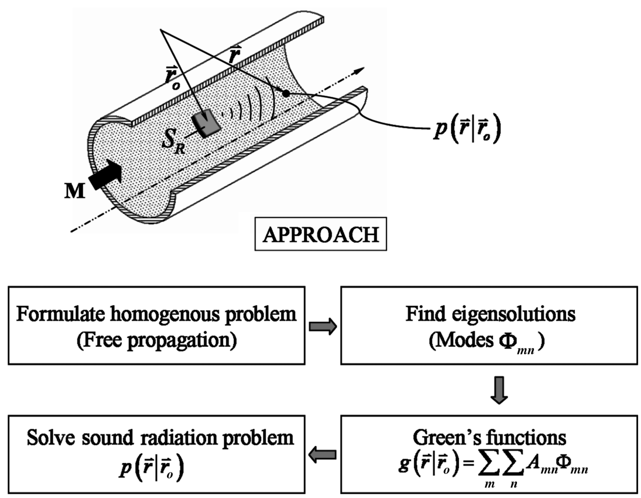

Fig. 1 Schematic of sound radiation formulation approach.

This approach leads one to express the solution in closed form as a series expansion of the obtained eigenfunctions. The solution is completely defined once the coefficients A_{mn} of the modal series are computed appropriately. At this point, the method allows one to explicitly impose the continuity condition at the source plane to obtain an expression to solve for these modal amplitudes. It is important to remark that this condition is mandatory for the existence of the Green's function of a second-order differential equation, that is, the wave equation. Later, the solution is extended to the radiation of sound from a finite portion of moving boundary by using the divergence theorem. The systematic application of this theorem allows satisfying the soft-wall boundary condition as well as including the effect of the flow. The following sections present the steps described in detail.

III. Eigenvalue Problem in a Lined Circular Duct with Flow

The derivation of the spectral expansion requires investigating the solution of the homogeneous convected wave equation in cylindrical coordinates in terms of acoustic modes [3]. Therefore, the acoustic pressure inside the lined circular duct is considered as a linear superposition of propagating modes as

$$p_\omega(\boldsymbol{r}) = \sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}^{(+)}\Phi_{mn}^{(+)}(r,\theta)e^{-ik_z^{(+)}z} + \sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}^{(-)}\Phi_{mn}^{(-)}(r,\theta)e^{-ik_z^{(-)}z} \tag{2}$$

where the superscripts $(+)$ and $(-)$ indicate variables associated to positive and negative z-direction propagation, respectively. The subscripts m and n refer to the circumferential and radial mode order, respectively. To find the expressions for the acoustic modes, the solution to the wave equation is assumed to be harmonic of the form $e^{i\omega t}$, separable, and propagating in the infinite z direction as

$$p(r,\theta,z,t) = \Phi(r,\theta)e^{-ik_z z}e^{i\omega t} \tag{3}$$

where k_z is the propagating constant, that is, axial wave number, and the solution $\Phi(r,\theta)$ needs to satisfy the soft-wall boundary condition. Note that the time-dependent term $e^{i\omega t}$ will not be shown in the remaining derivations. The solution $\Phi(r,\theta)$ is also assumed to be separable as the product of two functions as follows:

$$\Phi(r,\theta) = \Theta(\theta)\cdot R(r) \tag{4}$$

where $\Theta(\theta)$ satisfies the periodic condition

$$\Theta(\theta) = \Theta(\theta + 2m\pi) \qquad m = 0, 1, 2, \ldots \tag{5}$$

Because the cylindrical system of coordinates is used, the expressions for the solutions $\Theta(\theta)$ and $R(r)$ are well determined for the case of a circular duct. The circumferential solution corresponds to a harmonic oscillator and can be represented by a sine, cosine, or complex exponential functions. For convenience, this work will adopt the cosine function to easily align the argument with respect to the source location. However, note that the final solution can be easily expressed in terms of exponential functions by using Euler's trigonometric formulas. Thus, the circumferential solution is chosen as

$$\Theta(\theta) = A\cos(m\theta) \qquad m = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{6}$$

Note that the argument reference phase θ_o is omitted only for the purpose of solving the eigenvalue problem. This reference phase will appear later in the derivation of the Green's function as the azimuthal location of the source. In addition, the radial solution $R(r)$ is the mth order first kind Bessel function as

$$R(r) = CJ_m(k_{mn}r) \qquad m = 1, 2, 3, \ldots, \qquad n = 1, 2, 3, \ldots \tag{7}$$

where A and C are arbitrary constants and the argument k_{mn} is the mode eigenvalue. Therefore, the modal solutions are expressed as

$$\Phi_{mn}(r,\theta) = A_{mn}\cos(m\theta)J_m(k_{mn}r) \tag{8}$$

where A_{mn} is a constant for every mode. Note that to completely express the solution, the propagating constant k_z for every mode is defined by the following dispersion relationship:

$$k_{mn}^2 = k_o^2 - k_z^2(1 - M^2) - 2k_o k_z M \tag{9}$$

where M is the flow Mach number, $M = U_z/c$ in the positive z direction with U_z being the actual flow speed and c the speed of sound. The term $k_o = \omega/c$ is the free field wave number.

The determination of the mode eigenvalues k_{mn} is based on the requirement to satisfy the duct wall boundary condition. This boundary condition needs to be expressed as a relation between the pressure at the wall and the pressure gradient normal to the wall. To this end, the Euler's equilibrium equation in the direction normal to the wall is considered as follows:

$$-\frac{\partial p}{\partial r} = \rho\left(\frac{\partial}{\partial t} + U_z\frac{\partial}{\partial z}\right)v_r \tag{10}$$

where U_z is again the convective uniform flow velocity and $v_r(r,\theta,z)$ is the acoustic particle velocity in the r direction, which is assumed to also take the form $v_r(r,\theta,z) = v_r(r,\theta)e^{-ik_z z}$. Therefore, the momentum equation in the r direction evaluated at the wall becomes

$$-\left.\frac{\partial p}{\partial r}\right|_{r=a} = i\rho c(k_o - k_z M)v_r|_{r=a} \tag{11}$$

Then, the particle velocity at the wall $v_r(a,\theta)$ is obtained by matching the acoustic particle displacement normal to the wall in the neighborhood of the liner surface, as described by Ko [6]. In the limit of vanishing viscosity, the boundary layer in this region reduces to an infinitely thin vortex sheet that separates a region without flow (inside liner) and the acoustic field in the flow [14]. This scenario is illustrated in Fig. 2. In the liner side, the air particles move perpendicular to the liner surface, whereas, in the duct side, they are also convected by the flow. Then, as the kinematic effect of this shear layer must be introduced in the formulation of the boundary condition, a discontinuity in the normal particle velocity appears and the correct boundary condition is obtained by using the normal particle displacement. This yields

$$\frac{v^\ell}{i\omega} = \frac{v_r}{i\omega[1 - M(k_z/k_o)]} = \frac{v_r}{i\omega(k_o - Mk_z)/k_o} \tag{12}$$

where v^ℓ is the particle velocity in the liner.

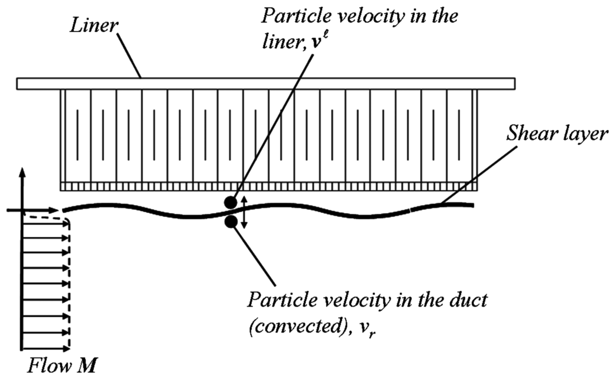

Fig. 2 Particle velocities inside and outside the shear layer.

In addition, the soft-wall properties of a locally reactive liner are characterized by the impedance Z_w. The impedance is commonly written in terms of the specific admittance β_w as

$$Z_w = \frac{\rho c}{\beta_w} \tag{13}$$

where ρ is the fluid density. The particle velocity and the pressure in the liner surface are related through the "locally reacting" liner specific admittance as

$$v^\ell = \frac{p^\ell}{Z_w} = \beta_w \frac{p^\ell}{\rho c} \tag{14}$$

where p^ℓ is the pressure in the liner. Then, because the pressure across the shear layer remains constant, that is, $p = p^\ell$, the combination of Eq. (12) with Eq. (14) leads to obtain the particle velocity at the wall as

$$v_r|_{r=a} = \frac{\beta_w}{\rho c}\frac{(k_o - k_z M)}{k_o} p|_{r=a} \tag{15}$$

Finally, the boundary condition for the pressure is obtained by replacing Eq. (15) into Eq. (11) as follows [10,11]:

$$\left.\frac{\partial p}{\partial r}\right|_{r=a} = -\, i\beta_w \frac{(k_o - k_z M)^2}{k_o} p|_{r=a} \tag{16}$$

An alternative approach to obtain this equation would be to use more of the general formulation using the boundary condition derived by Myers [9]. Nevertheless, for the case of inviscid flow and constant cross-section ducts, this formulation leads one to obtain the same expression for Eq. (16).

To solve for the mode eigenvalues k_{mn}, the pressure in Eq. (16) is considered as defined in Eq. (3) with the mode shape as in Eq. (8). This leads to

$$i\beta_w \frac{(k_o - Mk_z)^2}{k_o} J_m(k_{mn}a) = -k_{mn}J'_m(k_{mn}a) \tag{17}$$

In the presence of flow, that is, $M \neq 0$, this characteristic equation depends on the value of the axial wave number k_z, which introduces certain difficulty to the problem. In fact, it was noted previously that the eigenvalues k_{mn} and the propagation constants k_z are related by the quadratic relation in Eq. (9). Therefore, an eigenvalue k_{mn} yields two values of k_z that correspond to each direction of propagation. The two values for k_z are

$$\begin{aligned} k_z^{(+)} &= \frac{-k_o M + \sqrt{k_o^2 - (1-M^2)k_{mn}^2}}{(1-M^2)} \\ k_z^{(-)} &= \frac{-k_o M - \sqrt{k_o^2 - (1-M^2)k_{mn}^2}}{(1-M^2)} \end{aligned} \tag{18}$$

where $k_z^{(+)}$ and $k_z^{(-)}$ correspond to positive and negative z-direction propagating waves. This implies that the value of k_{mn} and $J_m(\cdot)$ will also depend on the direction of wave propagation. Thus, Eq. (17) has to be solved for both positive and negative propagation directions as

$$i\beta_w \frac{(k_o - Mk_z^{(+)})^2}{k_o} J_m(k_{mn}^{(+)}a) = -k_{mn}^{(+)}J'_m(k_{mn}^{(+)}a) \tag{19a}$$

where $k_z^{(+)}$ is obtained by replacing $k_{mn}^{(+)}$ into the first relation in Eq. (18), and

$$i\beta_w \frac{(k_o - Mk_z^{(-)})^2}{k_o} J_m(k_{mn}^{(-)}a) = -k_{mn}^{(-)}J'_m(k_{mn}^{(-)}a) \tag{19b}$$

where $k_z^{(-)}$ is obtained by replacing $k_{mn}^{(-)}$ into the second relation in Eq. (18). Finally, the solution for the acoustic modes is

$$\begin{aligned} \Phi_{mn}^{(+)}(r,\theta) &= A_{mn}^{(+)}\cos(m\theta)J_m(k_{mn}^{(+)}r) \\ \Phi_{mn}^{(-)}(r,\theta) &= A_{mn}^{(-)}\cos(m\theta)J_m(k_{mn}^{(-)}r) \end{aligned} \tag{20}$$

for positive and negative z-direction propagations, respectively. The constants $A_{mn}^{(+)}$ and $A_{mn}^{(-)}$ are defined as the modal amplitudes. The values $k_{mn}^{(+)}$ and $k_{mn}^{(-)}$ are the mode wave numbers for the positive and negative propagating modes, respectively.

The numerical solution for the eigenvalues is still a subject of investigation given the difficulty of solving the characteristic equations in Eqs. (19a) and (19b). Because these characteristic equations depend on the wall impedance, the eigenvalues $k_{mn}^{(+)}$ and $k_{mn}^{(-)}$ are found to be complex and dependent upon frequency. The solving technique implemented on the numerical examples in this paper is based on tracking the eigenvalues from low frequencies, where the liner impedance behaves as the hard-wall condition. In addition, the zeros of the characteristic equations are found by using a minimization technique, namely, the Nelder–Mead simplex method.

An interesting result found by Tester [7] on the eigenvalue solution is the existence of certain strange modes which have been defined as having phase speed in the opposite direction to that of attenuation. These strange modes were later investigated in detail by Rienstra [14] and were identified to represent surface waves meaning that the corresponding mode is spatially confined to the immediate neighborhood of the wall. A strict causality analysis suggested that these strange surface waves could be representing spatial instabilities for certain combinations of impedance and Mach numbers, and they might be associated with the unstable nature of the wall shear layer. Several other references [15–19] recognize the possibility of spatial instabilities, but their existence is still very much an open question because there is no experimental evidence of spatial amplification on real systems. In fact, some investigators solving the mode propagation in finite length liners never mentioned the possibility of any unstable mode [11,20,21]. A study presented by Koch and Möhring [22] remarks that the presence of instabilities may be compensated by other nonlinearities which are, in fact, not modeled. Thus, these instabilities must be disregarded in order to obtain a first approximation with a linear model. In addition, Tester [7] also showed that the instabilities that could appear in lined ducts with flow are of the convective type and would not affect the validity of a steady-state mode series representation of the acoustic field inside the duct. In the present work, the derivation of the Green's functions will be performed using a mode series solution without considering the possibility of any instability. Therefore, all modes will be assumed to decay in the same direction of propagation.

IV. Point Source: The Exact Green's Functions

The acoustic field Green's function for a point source inside the lined duct is found as the solution of the nonhomogenous convected wave equation subjected to the soft-wall boundary condition. The utilized method to solve the problem is based on assuming the solution to be a linear combination of the modes present inside the duct, that is, *modal expansion approach*. The application of this method leads one to find an exact closed-form expression for the solution that satisfies the continuity condition at the source location.

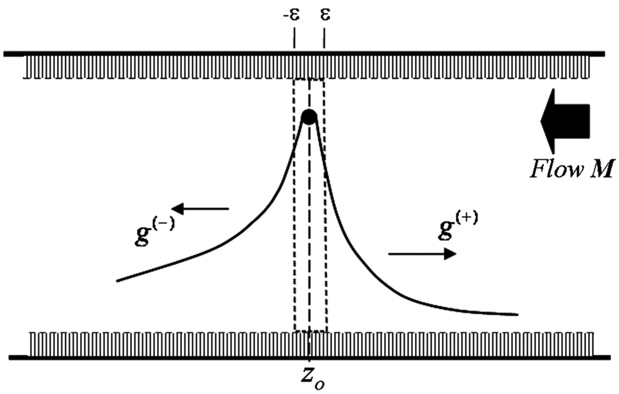

Fig. 3 Solution of Green's function in the positive and negative propagating fields: condition of continuity in a small volume near the source at z_0.

To model the presence of the point source, the nonhomogeneous (Laplace transformed) convected wave equation is first obtained by adding a Dirac delta function as an input term in the right-hand side of the equation. In cylindrical coordinates, this equation becomes

$$\frac{\partial^2 g}{\partial r^2}+\frac{1}{r}\frac{\partial g}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2}+\frac{\partial^2 g}{\partial z^2}(1-M^2)-2iMk_o\frac{\partial g}{\partial z}+k_o^2 g = \frac{1}{r}\delta(\boldsymbol{r}\mid \boldsymbol{r}_0) \tag{21}$$

and the soft-wall boundary condition [11]:

$$\left.\frac{\partial g}{\partial r}\right|_{r=a}=-i\beta_w\frac{(k_o-k_z M)^2}{k_o}g \tag{22}$$

where g is the sought Green's function, $\boldsymbol{r}_0=(r_0,\theta_0,z_0)$ is the location of the simple source, and $\delta(\boldsymbol{r}\mid\boldsymbol{r}_0)=\delta(r-r_0)\delta(\theta-\theta_0)\delta(z-z_0)$ is the delta Dirac function.

The solution to Eqs. (21) and (22) is assumed as a linear combination of the acoustic modes inside the duct. In the case of the hard-wall duct, the mode shapes are independent of the direction of propagation which results in a straightforward formulation that leads to a well-known solution [1,2,7,23]. On the contrary, according to the expressions in Eq. (20), the acoustic modes in the soft-wall duct with flow have different expressions depending on the direction of propagation. Figure 3 illustrates the approach that requires expressing the solution as two functions depending on the propagation direction, that is, $g^{(+)}$ and $g^{(-)}$. Therefore, depending on the relative location between the observer and the point source at z_0, the Green's function is expressed as follows

$$\begin{aligned} g^{(+)}(\boldsymbol{r}\mid\boldsymbol{r}_0)&=\sum_{m=0}^{M}\sum_{n=0}^{N}A_{mn}^{(+)}\Phi_{mn}^{(+)}e^{-ik_z^{(+)}(z-z_0)} \quad z>z_0\\ g^{(-)}(\boldsymbol{r}\mid\boldsymbol{r}_0)&=\sum_{m=0}^{M}\sum_{n=0}^{N}A_{mn}^{(-)}\Phi_{mn}^{(-)}e^{-ik_z^{(-)}(z-z_0)} \quad z<z_0 \end{aligned} \tag{23}$$

where M and N are the maximum number of terms to be included in the expansion, that is, number of acoustic modes to use in the expansion.

A critical step in the derivation is to define the sought solution as a single continuous function in the spatial domain. To this end, the Green's functions are expressed with the help of the Heaviside function $H(z-z_0)$ [i.e., $H(z)=1$ $(z>0)$; $H(z)=0$ $(z<0)$, and $H(0)=1/2$] as follows:

$$g(\boldsymbol{r}\mid\boldsymbol{r}_0)=g^{(+)}(\boldsymbol{r}\mid\boldsymbol{r}_0)H(z-z_0)+g^{(-)}(\boldsymbol{r}\mid\boldsymbol{r}_0)[1-H(z-z_0)] \tag{24}$$

For the Green's functions, the mode shapes are selected to have the form

$$\begin{aligned} \Phi_{mn}^{(+)}(r,\theta)&=\cos[m(\theta-\theta_0)]J_m(k_{mn}^{(+)}r)\\ \Phi_{mn}^{(-)}(r,\theta)&=\cos[m(\theta-\theta_0)]J_m(k_{mn}^{(-)}r) \end{aligned} \tag{25}$$

which already consider the reference with respect to the circumferential location of the source. The problem is now reduced to find the modal amplitudes $A_{mn}^{(+)}$ and $A_{mn}^{(-)}$ that define the Green's functions in Eq. (23). This task involves the use of two required conditions: 1) discontinuity in the slope of the Green's function and 2) continuity of the Green's function at the source location $z=z_0$. The two conditions will be described in the following subsections.

A. Discontinuity in the Slope of the Green's Function

The first condition is to impose the discontinuity in the slope at the point source location by use of the Dirac delta function. For the sake of clarity, a compact notation is used for the linear operator

$$L(\,\cdot\,)=\frac{\partial^2(\,\cdot\,)}{\partial r^2}+\frac{1}{r}\frac{\partial(\,\cdot\,)}{\partial r}+\frac{1}{r^2}\frac{\partial^2(\,\cdot\,)}{\partial\theta^2}$$

which does not depend on the z coordinate. Then, Eq. (21) can be rewritten as

$$k_o^2 g+L(g)+\frac{\partial^2 g}{\partial z^2}(1-M^2)-2iMk_o\frac{\partial g}{\partial z} = \delta(r-r_0)\frac{\delta(\theta-\theta_0)}{r}\delta(z-z_0) \tag{26}$$

Replacing Eq. (24) into Eq. (26) yields

$$\begin{aligned} &(k_o^2+L)\{g^{(+)}H(z-z_0)+g^{(-)}[1-H(z-z_0)]\}\\ &+(1-M^2)\frac{\partial^2}{\partial z^2}\{g^{(+)}H(z-z_0)+g^{(-)}[1-H(z-z_0)]\}\\ &-2iMk_o\frac{\partial}{\partial z}\{g^{(+)}H(z-z_0)+g^{(-)}[1-H(z-z_0)]\}\\ &=\frac{1}{r}\delta(\boldsymbol{r}\mid\boldsymbol{r}_0) \end{aligned} \tag{27}$$

Considering that the derivative of the Heaviside function is $H'(z-z_0)=\delta(z-z_0)$, Eq. (27) can be rearranged as follows:

$$\begin{aligned} &(k_o^2+L)\{g^{(+)}H(z-z_0)+g^{(-)}[1-H(z-z_0)]\}\\ &+(1-M^2)\left[\frac{\partial^2 g^{(+)}}{\partial z^2}H(z-z_0)+\frac{\partial^2 g^{(-)}}{\partial z^2}[1-H(z-z_0)]\right.\\ &\left.+2\left(\frac{\partial g^{(+)}}{\partial z}-\frac{\partial g^{(-)}}{\partial z}\right)\delta(z-z_0)+(g^{(+)}-g^{(-)})\delta'(z-z_0)\right]\\ &-2iMk_o\left[\frac{\partial g^{(+)}}{\partial z}H(z-z_0)+\frac{\partial g^{(-)}}{\partial z}[1-H(z-z_0)]\right.\\ &\left.+(g^{(+)}-g^{(-)})\delta(z-z_0)\right]=\frac{1}{r}\delta(\boldsymbol{r}\mid\boldsymbol{r}_0) \end{aligned} \tag{28}$$

To find the modal amplitudes $A_{mn}^{(+)}$ and $A_{mn}^{(-)}$, Eq. (28) needs to be premultiplied by the acoustic modes defined in the complete spatial domain as follows:

$$\Phi_{er}=\Phi_{er}^{(+)}H(z-z_0)+\Phi_{er}^{(-)}[1-H(z-z_0)]$$

and integrated over the small volume shown in Fig. 3. The axial dimension of this volume is defined as 2ε, where $\varepsilon\to 0$. After imposing continuity (*second condition*), solving the integral in the z coordinate, and taking the limit $\varepsilon\to 0$, it can be shown that most terms in Eq. (28) vanish. This procedure leads to

$$\int_0^a \int_0^{2\pi} \lim_{\varepsilon\to 0} \int_{z_o-\varepsilon}^{z_o+\varepsilon} (1-M)\left\{\left(\frac{\partial g^{(+)}}{\partial z} - \frac{\partial g^{(-)}}{\partial z}\right)\{\Phi_{er}^{(+)} H(z-z_0) + \Phi_{er}^{(-)}[1-H(z-z_0)]\}\delta(z-z_0)\right\} \mathrm{d}z r\,\mathrm{d}\theta\,\mathrm{d}r$$
$$= \int_0^a \int_0^{2\pi} \lim_{\varepsilon\to 0} \int_{z_0-\varepsilon}^{z_0+\varepsilon} \left\{\delta(r-r_0)\frac{\delta(\theta-\theta_0)}{\not r}\delta(z-z_0) \times \{\Phi_{er}^{(+)} H(z-z_0) + \Phi_{er}^{(-)}[1-H(z-z_0)]\}\right\} \mathrm{d}z \not r\,\mathrm{d}\theta\,\mathrm{d}r \tag{29}$$

Finally, replacing the expanded solution in Eq. (23) into Eq. (29) and solving the integrals yields the following set of $M \times N$ equations:

$$\sum_{n=0}^{N}\left\{k_z^{(+)} A_{mn}^{(+)} \int_0^{2\pi}\int_0^a \left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(+)} r\,\mathrm{d}\theta\,\mathrm{d}r - k_z^{(-)} A_{mn}^{(-)} \int_0^{2\pi}\int_0^a \left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(-)} r\,\mathrm{d}\theta\,\mathrm{d}r\right\}$$
$$= i\,\frac{\Phi_{mr}^{(+)}(r_0,\theta_0)+\Phi_{mr}^{(-)}(r_0,\theta_0)}{2(1-M^2)}$$
$$m = 0,1,2,3,\ldots,M; \qquad r = 0,1,2,3,\ldots,N \tag{30}$$

where the factor 2 in the denominator of the right-hand side appears as a consequence of solving the Dirac delta integral on the axial location of the source. Note that the orthogonality of the modes in the circumferential direction has been used in Eq. (30). However, the system of equations is fully coupled because the modes are not orthogonal in the radial direction. It is important to note that the mathematical derivations to arrive at Eq. (30) are very extensive and complex. A more indepth description of these derivations can be found in the work by Alonso [24].

B. Continuity of the Green's Function

Equation (30) is a system of $M \times N$ equations with $2(M \times N)$ unknowns, that is, the modal amplitudes $A_{mn}^{(+)}$ plus the $A_{mn}^{(-)}$. This is again because the sound structure propagating in the positive direction is different from the negative propagating field. The remaining set of $M \times N$ equations is obtained by explicitly imposing the continuity of the Green's function at $z = z_0$ as follows [25]:

$$g^{(+)}|_{z=z_0} = g^{(-)}|_{z=z_0} \tag{31}$$

It is important to remark that the formulation by Zorumski [11] does not explicitly impose the continuity condition as the formulation in Eq. (31). To develop this condition, Eq. (31) is simply premultiplied by $(\Phi_{er}^{(+)} + \Phi_{er}^{(-)})/2$ and integrated over the duct cross section to yield

$$\sum_{n=0}^{N}\left\{A_{mn}^{(+)} \int_0^{2\pi}\int_0^a \left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(+)} r\,\mathrm{d}\theta\,\mathrm{d}r - A_{mn}^{(-)} \int_0^{2\pi}\int_0^a \left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(-)} r\,\mathrm{d}\theta\,\mathrm{d}r\right\} = 0$$
$$m = 0,1,2,3,\ldots \tag{32}$$

As with the system in Eq. (30), this system of equations is fully coupled because the modes are not orthogonal in the radial coordinate.

C. Modal Amplitudes

This section summarizes the computation of the modal amplitudes $A_{mn}^{(+)}$ and $A_{mn}^{(-)}$. The system of equations in Eqs. (30) and (32) can be grouped and written in matrix form as

$$\begin{bmatrix} [\Lambda_{m,nr}^{(+)}][k_z^{(+)}] & -[\Lambda_{m,nr}^{(-)}][k_z^{(-)}] \\ [\Lambda_{m,nr}^{(+)}] & -[\Lambda_{m,nr}^{(-)}] \end{bmatrix} \begin{Bmatrix} A_{mn}^{(+)} \\ A_{mn}^{(-)} \end{Bmatrix} = \begin{Bmatrix} \Psi_r \\ \underline{0} \end{Bmatrix} \qquad m = 0,1,2,3,\ldots \tag{33}$$

where

$$\{\Psi_r\} = i\,\frac{[\Phi_{mr}^{(+)}(r_0,\theta_0)+\Phi_{mr}^{(-)}(r_0,\theta_0)]}{2\pi a^2(1-M^2)} \tag{34}$$

The matrices $[\Lambda_{m,nr}^{(+)}]$ and $[\Lambda_{m,nr}^{(-)}]$ are fully populated because the modes are not orthogonal in a lined duct with flow, and their components are defined by

$$\Lambda_{m,nr}^{(+)} = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a \left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(+)} r\,\mathrm{d}\theta\,\mathrm{d}r$$
$$\Lambda_{m,nr}^{(-)} = \frac{1}{\pi a^2}\int_0^{2\pi}\int_0^a \left(\frac{\Phi_{mr}^{(+)}+\Phi_{mr}^{(-)}}{2}\right)\Phi_{mn}^{(-)} r\,\mathrm{d}\theta\,\mathrm{d}r \tag{35}$$

On the other hand, matrices $[k_z^{(+)}]$ and $[k_z^{(-)}]$ are diagonal and contain the axial wave number for each mode. Note that Eq. (30) was divided by πa^2 and thus this term appears in the denominator of Eqs. (34) and (35).

In the particular case of the hard-wall duct, the formulation simplifies because the modes are orthogonal to each other and do not depend on the direction of propagation. For this reason, the normalization factors in Eq. (35) collapse in the same expressions, which also vanish for $r \neq n$. Consequently, the system of equations in Eq. (33) becomes diagonal, leading to a direct formula for the Green's function modal amplitudes.

Thus, the Green's function is then computed by solving the linear system of equations in Eq. (33) for the modal amplitudes that in turn are then used in Eq. (23).

D. Adjoint Solution

The found Green's function can now be used to investigate the sound radiation from a finite piston source. The extension of the solution to the finite region is performed using the divergence theorem that leads to integrate the Green's function over the radiating surface. In addition, this process involves applying the Maxwell reciprocity principle to correctly reference the radiated pressure to the location of the source [26,27]. However, the operator that defines the convected wave equation in Eq. (21) with the soft-wall boundary condition in Eq. (22) is not self-adjoint. The reason for this is due to both the presence of the flow and the nonhomogeneous boundary condition given by $\beta_w \neq 0$. As a consequence, the Green's function derived in this section does not directly satisfy the Maxwell reciprocity principle, that is, $g(\boldsymbol{r} \mid \boldsymbol{r}_0) \neq g(\boldsymbol{r}_0 \mid \boldsymbol{r})$. In this case, the Maxwell reciprocity has to be defined in terms of the adjoint Green's function which is the solution of the associated adjoint acoustic problem as will be shown. To this end, it is required to consider a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ defined as follows:

$$\langle f, h\rangle = \int_{-\infty}^{\infty}\int_0^a\int_0^{2\pi} f(r,\theta,z)\cdot\overline{h(r,\theta,z)}\,\mathrm{d}\theta r\,\mathrm{d}r\,\mathrm{d}z \tag{36}$$

where the bar $(\bar{\cdot})$ denotes the complex conjugate. Then, if the Green's function g is assumed to belong in this Hilbert space, the associated adjoint problem for the convected acoustic wave equation in a lined duct is given as follows [3,11,26]:

$$\frac{\partial^2 g^*}{\partial r^2} + \frac{1}{r}\frac{\partial g^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2 g^*}{\partial\theta^2} + \frac{\partial^2 g^*}{\partial z^2}(1-M^2) - 2iMk_o\frac{\partial g^*}{\partial z} + k_o^2 g^* = \frac{1}{r}\delta(\boldsymbol{r}|\boldsymbol{r}_0) \tag{37}$$

under the soft-wall boundary condition [10,11]:

$$\overline{\frac{\partial g^*}{\partial r}}\Bigg|_{r=a} = -i\beta_w \frac{(k_o - k_z M)^2}{k_o}\overline{g^*} \tag{38}$$

where g^* is the adjoint Green's function and the bar over a variable, for example, $\bar{f}$, indicates the complex conjugate. Equations (37) and (38) show that the difference between the direct and the adjoint problem of the convected wave equation is given only by the boundary condition, that is, the boundary condition on the adjoint problem is given in terms of the complex conjugate of the solution. Consequently, the problem defined by Eqs. (37) and (38) has a similar analytical solution as the direct problem defined by Eqs. (21) and (22). The difference is given by the eigenvalues corresponding to the new boundary condition defined in Eq. (38). The eigenfunctions can be found indeed following the same steps as from Eqs. (3–20). Finally, they are selected to have the form

$$\begin{aligned}\Phi_{mn}^{*(+)}(r,\theta) &= \cos[m(\theta-\theta_0)]J_m(k_{mn}^{*(+)}r)\\ \Phi_{mn}^{*(-)}(r,\theta) &= \cos[m(\theta-\theta_0)]J_m(k_{mn}^{*(-)}r)\end{aligned} \tag{39}$$

The eigenvalues $k_{mn}^{*(+)}$ and $k_{mn}^{*(-)}$ are obtained from replacing the eigenfunctions into the boundary condition as follows:

$$-i\beta_w \frac{(k_o - Mk_z^{*(+)})^2}{k_o}\overline{J_m(k_{mn}^{*(+)}a)} = \overline{k_{mn}^{*(+)}J'_m(k_{mn}^{*(+)}a)} \tag{40a}$$

$$-i\beta_w \frac{(k_o - Mk_z^{*(-)})^2}{k_o}\overline{J_m(k_{mn}^{*(-)}a)} = \overline{k_{mn}^{*(-)}J'_m(k_{mn}^{*(-)}a)} \tag{40b}$$

where the relation between k_{mn}^* and k_z^* is now given by

$$k_{mn}^{*2} = k_o^2 - k_z^{*2}(1-M^2) - 2k_o k_z^* M \tag{41}$$

At this point, it is clear that solving the characteristic equations in Eqs. (40a) and (40b) in combination with Eq. (41) yields the complex conjugate of the eigenvalues of the direct problem. Then, the eigenvalues of the associated adjoint problem can be related to the direct problem as follows:

$$k_{mn}^{*(+)} = \overline{k_{mn}^{(+)}}; \qquad k_{mn}^{*(-)} = \overline{k_{mn}^{(-)}} \tag{42}$$

Also, the axial wave numbers are given by

$$k_z^{*(+)} = \overline{k_z^{(+)}}; \qquad k_z^{*(-)} = \overline{k_z^{(-)}} \tag{43}$$

In addition, the modal amplitudes $A_{mn}^{*(+)}$ and $A_{mn}^{*(-)}$ defining the adjoint Green's function can be found following the same steps presented from Eqs. (23–35). This procedure will also lead one to find the modal amplitudes of the adjoint Green's function in terms of the direct solution as

$$A_{mn}^{*(+)} = \overline{A_{mn}^{(+)}}; \qquad A_{mn}^{*(-)} = \overline{A_{mn}^{(-)}} \tag{44}$$

Finally, the associated adjoint Green's function is then expressed as

$$\begin{aligned}g^{*(+)}(\boldsymbol{r}\mid\boldsymbol{r}_0) &= \sum_{m=0}^{M}\sum_{n=0}^{N}\overline{A_{mn}^{(+)}}\,\overline{\Phi_{mn}^{(+)}}\,e^{-i\overline{k_z^{(+)}}(z-z_0)}\\ g^{*(-)}(\boldsymbol{r}\mid\boldsymbol{r}_0) &= \sum_{m=0}^{M}\sum_{n=0}^{N}\overline{A_{mn}^{(-)}}\,\overline{\Phi_{mn}^{(-)}}\,e^{-i\overline{k_z^{(-)}}(z-z_0)}\end{aligned} \tag{45}$$

The direction of propagation of the solutions $g^{*(+)}$ and $g^{*(-)}$ can be determined based on the condition that the mode amplitudes must be bounded, that is, to have decaying propagating modes. The propagating properties of every mode are given by the exponential term of Eq. (45), which, in fact, depends on the complex conjugate of the axial wave number of the direct problem, that is, $\overline{k_z^{(+)}}$ and $\overline{k_z^{(-)}}$. Consequently, the solutions $g^{*(+)}$ and $g^{*(-)}$ are bounded only if the propagating directions are assigned opposite to the direct problem as

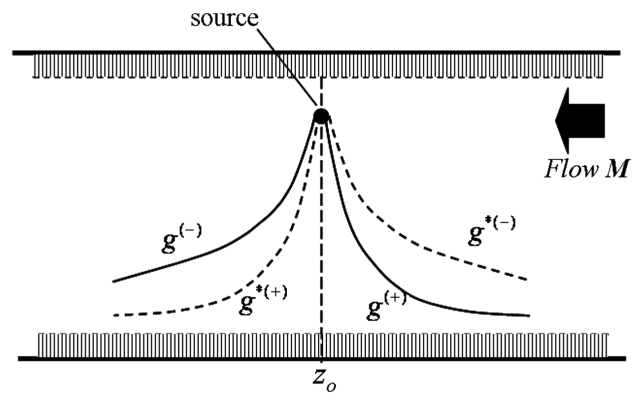

Fig. 4 Schematic of the adjoint Green's functions propagating direction.

$$g^{*(+)} \quad \text{for } z < z_0 \qquad g^{*(-)} \quad \text{for } z > z_0 \tag{46}$$

Figure 4 presents a schematic of the propagating direction of the Green's functions. The solid line represents the Green's function for the direct problem while the dotted line corresponds to the adjoint solution. Note that there is a symmetry between the positive functions $g^{(+)}$ and $g^{*(+)}$, as well as between the negative solutions $g^{(-)}$ and $g^{*(-)}$. This symmetry yields the Maxwell reciprocity [26,27], which in this case is satisfied in terms of the adjoint solution in Eq. (45) rather than by simply reversing the source position as for typical self-adjoint problems.

To formally state the reciprocity, it is convenient to express the associated adjoint Green's function for the complete domain in terms of the Heaviside function, that is, similarly to the direct solution. That is

$$g^*(\boldsymbol{r}\mid\boldsymbol{r}_0) = g^{*(+)}(\boldsymbol{r}\mid\boldsymbol{r}_0)[1-H(z-z_0)] + g^{*(-)}(\boldsymbol{r}\mid\boldsymbol{r}_0)H(z-z_0) \tag{47}$$

Then, the reciprocity principle, which will be used in the finite piston formulation, is satisfied in general as follows:

$$g(\boldsymbol{r}\mid\boldsymbol{r}_0) = \overline{g^*(\boldsymbol{r}_0\mid\boldsymbol{r})} \tag{48}$$

V. Finite Piston Source

The Green's functions can now be used to find the sound field due to a motion of a finite piston source vibrating with known uniform velocity V_p. Figure 5 illustrates a sector of the soft wall in the circular duct with the presence of the vibrating piston. For the sake of clarity, this piston source is referred to as the radiating surface of the duct. The rest of the duct surface is referred to as a nonradiating surface. The derivation in this section follows the approach taken by Morse and Ingard [3] for radiation from boundaries of the lined duct without flow. However, the formulation is here extended to the case with flow.

It is convenient to first analyze the behavior of the nonradiating surface, that is, liner. As shown in Fig. 5, the surface of the liner is covered by a shear layer which separates the flow region from the liner's physical surface. Although the nonradiating areas of the duct boundary do not move, this shear layer is allowed to have a radial movement given by the permeability characteristics of the soft wall. Therefore, the radial particle velocity v_r at the lined wall does not vanish. The relationship between this particle velocity v_r (inside the flow region) and the pressure at the wall surface was derived in Eq. (15). This expression depends on the axial wave numbers k_z, which are certainly a property of every mode. Therefore, the behavior of the particle velocity v_r needs to be investigated in terms of its modal components $v_{r_{mn}}$. The modal particle velocity $v_{r_{mn}}$ in the radial direction is then related to the modal pressure p_{mn} over the liner (nonradiating surface) in the flow as follows [3]:

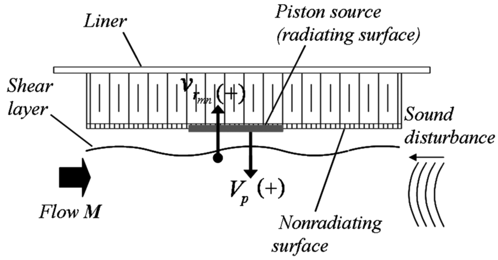

Fig. 5 Schematic of the shear layer at the lined duct walls. The positive convention of particle velocity v_r is outward. The positive convention of piston velocity V_p is inward.

$$v_{r_{mn}} = -\frac{1}{i\rho c(k_o - k_z M)}\frac{\partial p_{mn}}{\partial r}\bigg|_{r=a} = \frac{\beta_\omega}{\rho c}\frac{(k_o - k_z M)}{k_o} p_{mn}\bigg|_{r=a} \tag{49}$$

The first two terms of the equality in Eq. (49) correspond to the equilibrium relation in Eq. (11) between the pressure gradient and the particle velocity, that is, Euler's equation. The last two terms relate the pressure and its gradient using the definition of the liner specific acoustic admittance as in Eq. (15). Then, the difference between the radial particle velocity and the term $(1/\rho c k_o)\beta_\omega(k_o - k_z M)p_{mn}|_{r=a}$ must vanish as follows:

$$v_{r_{mn}} - \frac{\beta_\omega}{\rho c}\frac{(k_o - k_z M)}{k_o} p_{mn}\bigg|_{r=a} = 0 \tag{50}$$

In the radiating areas, this difference cannot vanish because of the presence of the piston motion. This difference has to be the perturbation velocity in the flow produced by the piston motion, which is defined as

$$v_{d_{mn}} = -\left(\frac{1}{i\rho c(k_o - k_z M)}\frac{\partial p_{mn}}{\partial r} + \frac{\beta_w}{\rho c}\frac{(k_o - k_z M)}{k_o} p_{mn}\right)\bigg|_{r=a} \tag{51}$$

As shown in Fig. 5, the modal particle velocity in the flow due to the radiating piston $v_{d_{mn}}$ needs to be related to the piston velocity at the wall, that is, outside of the shear layer or in the physical surface of the liner. Applying particle displacement continuity [6], the relation between the piston velocity and the radial particle velocity outside the shear layer (in the flow) is given by

$$v_{d_{mn}} = -\frac{(k_o - k_z M)}{k_o} V_p \tag{52}$$

where the negative sign is used to change the positive velocity convention to be inward.

The radiation from the piston source is now obtained by applying the Green's divergence theorem with the adjoint solution of the Green's function [26]. The reason for this is the fact that the linear operator L defining the convected wave equation with a soft-wall boundary condition is not self-adjoint. Then, although the direct Green's function is commonly used to solve this problem, the adjoint solution must be considered. The pressure anywhere in the duct can be found taking the inner product of p with Eq. (21) and the homogeneous convected wave equation with g^*. Then subtract the results as follows:

$$p = \langle p, \mathsf{L}g^*\rangle - \langle \mathsf{L}p, g^*\rangle \tag{53}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in the Hilbert space as defined in Eq. (36). Considering the Green's function that satisfies Eq. (38), the application of the divergence theorem on Eq. (53) will lead to integration over the duct surface as

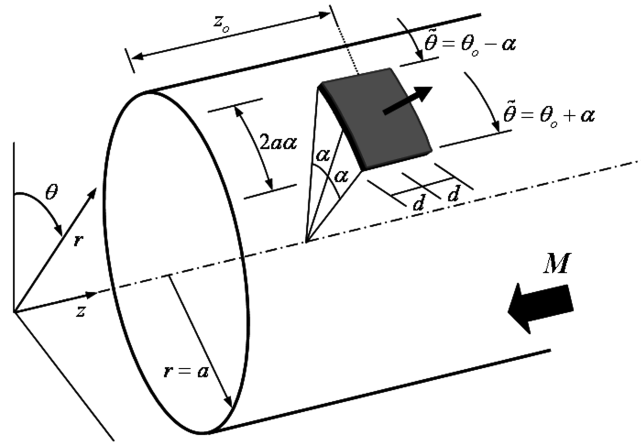

Fig. 6 Model of the finite sources.

$$\begin{aligned} p_{mn}(\boldsymbol{r}) &= -\int_S \left(\frac{\partial p_{mn}}{\partial r}\bigg|_{r=a}\overline{g^*_{mn}} - p_{mn}\overline{\frac{\partial g^*_{mn}}{\partial r}}\bigg|_{r=a}\right)\mathrm{d}S \\ &= -\int_S \left(\frac{\partial p_{mn}}{\partial r}\bigg|_{r=a}\overline{g^*_{mn}} + i p_{mn}\frac{\beta}{k_o}(k_o - k_z M)^2\overline{g^*_{mn}}\right)\mathrm{d}S \\ &= -i\rho c(k_o - k_z M)\int_S \left(\frac{1}{i\rho c(k_o - k_z M)}\frac{\partial p_{mn}}{\partial r}\right. \\ &\quad \left. + \frac{\beta}{\rho c}\frac{(k_o - k_z M)}{k_o} p_{mn}\right)\overline{g^*_{mn}}\,\mathrm{d}S \end{aligned} \tag{54}$$

As mentioned previously, only the radiating surface will be moving with velocity V_p. The integral over the nonradiating surface will then vanish because the Green's functions satisfy the soft-wall boundary condition. Thus, only the integral over the radiating surface, that is, piston source, needs to be solved. Then, replacing the factor in the integral of Eq. (54) by the perturbation velocity $v_{d_{mn}}$ in Eq. (51) leads to

$$p_{mn}(\boldsymbol{r}) = i\rho c(k_o - k_z M)v_{d_{mn}}\int_{S_R}\overline{g^*_{mn}(\boldsymbol{r}_0|\boldsymbol{r})}\,\mathrm{d}S \tag{55}$$

Note that in the last expression the integral is over the radiating surface S_R only, and because the piston velocity is uniform, $v_{d_{mn}}$ was taken outside the integral. Finally, using Eq. (52) and the Maxwell reciprocity stated in Eq. (48), the pressure inside the duct due to the motion of a finite source is given by

$$\begin{aligned} &p^{(+)}(\boldsymbol{r}\mid\boldsymbol{r}_0) \\ &\quad = -i\rho c V_p\sum_{m=0}^{M}\sum_{n=0}^{N}\frac{(k_o - k_z^{(+)}M)^2}{k_o}\int_{S_R} g^{(+)}_{mn}(\boldsymbol{r}\mid\boldsymbol{r}_0)\,\mathrm{d}S \quad z > z_0 \\ &p^{(-)}(\boldsymbol{r}\mid\boldsymbol{r}_0) \\ &\quad = -i\rho c V_p\sum_{m=0}^{M}\sum_{n=0}^{N}\frac{(k_o - k_z^{(-)}M)^2}{k_o}\int_{S_R} g^{(-)}_{mn}(\boldsymbol{r}|\boldsymbol{r}_0)\,\mathrm{d}S \quad z < z_0 \end{aligned} \tag{56}$$

Note that Eq. (56) is given for positive and negative directions of propagation, depending on the relative location between the source and the receiver.

For rectangular shapes, the integrals in equations can be solved in closed form. For the sake of completeness, this closed-form solution is presented next. The piston is assumed to be vibrating with a known uniform velocity V_p and it is defined by constant z and θ coordinates, as shown in Fig. 6. For convenience, the piston position is referring to the center point of the piston (a, θ_0, z_0) and its dimensions are given by the distance $\boldsymbol{d}$ and the angle $\boldsymbol{\alpha}$. In addition, the radiated sound field will be simultaneously derived for both directions of propagation given the similarity of the Green's functions expressions. For a point source located at a point $(a, \tilde{\theta}, \tilde{z})$ on the piston surface, the Green's function is given as

$$g^{(\ell)}(\boldsymbol{r} \mid \tilde{\boldsymbol{r}}) = \sum_{m=0}^{M_g} \sum_{n=0}^{N_g} A_{mn}^{(\ell)}(a) \cos[m(\theta - \tilde{\theta})] J_m(k_{mn}^{(\ell)} r) e^{-ik_z^{(\ell)}(z-\tilde{z})} \quad (57)$$

where $\ell = +$ is used for the sound field downstream of the source (positive z direction), while $\ell = -$ is used for the upstream field (negative z direction). Also, the modal amplitudes $A_{mn}^{(\ell)}$ are determined from Eq. (33).

To find the sound radiation formula, Eq. (57) is replaced into Eq. (56) and the resulting integral is solved over the piston surface as follows:

$$\int_{z_0-d}^{z_0+d} \int_{\theta_0-\alpha}^{\theta_0+\alpha} \cos[m(\theta - \tilde{\theta})] e^{-ik_z^{(\ell)}(z-\tilde{z})} a \, \mathrm{d}\tilde{\theta} \, \mathrm{d}\tilde{z}$$
$$= \frac{2a\alpha \sin(m\alpha)}{m\alpha} \cos m(\theta - \theta_0) e^{-ik_z^{(\ell)}(z-z_0)} \frac{\sin(k_z^{(\ell)} d)}{k_z^{(\ell)} d} 2d$$
$$= \kappa_\theta(\alpha) \cos m(\theta - \theta_0) e^{-ik_z^{(\ell)}(z-z_0)} \frac{\sin(k_z^{(\ell)} d)}{k_z^{(\ell)} d} 2d \quad (58)$$

where

$$\kappa_\theta(\alpha) = \frac{2a\alpha \sin(m\alpha)}{m\alpha}$$

For the special case of $m = 0$, this value is $\kappa_\theta(\alpha) = 2a\alpha$. Finally, this result yields the closed-form expression for the radiated sound field due to a rectangular piston as

$$p_{\text{piston}}^{(\ell)}(\boldsymbol{r} \mid \boldsymbol{r}_0) = -i\rho c V_p \sum_{m=0}^{M_g} \sum_{n=0}^{N_g} \frac{(k_o - k_z^{(\ell)} M)^2}{k_o} A_{mn}^{(\ell)}$$
$$\times \cos[m(\theta - \theta_0)] J_m(k_{mn}^{(\ell)} a) \boldsymbol{\kappa}_\theta(\alpha) e^{-ik_z^{(\ell)}(z-z_0)} \frac{\sin(k_z^{(\ell)} d)}{k_z^{(\ell)} d} 2d \quad (59)$$

Note that the obtained radiation formula is valid only when the observation point $\boldsymbol{r}$ is upstream or downstream of the source. If the axial location of the observation point is within the position of the piston, the integral in Eq. (58) has to be split into two regions, leading to a more complex result. In addition, the expression in Eq. (59) can be rewritten in a more practical fashion by using the concept of pressure transfer function. A pressure transfer function $Z^{(\ell)}(\boldsymbol{r} \mid \boldsymbol{r}_0)$ can be defined between the observation point and the rectangular piston vibrating with velocity V_p. This transfer function can be obtained from Eq. (59) by setting the piston velocity $V_p = 1$. Therefore, Eq. (59) can be expressed as

$$p_{\text{piston}}^{(\ell)}(\boldsymbol{r} \mid \boldsymbol{r}_0) = Z^{(\ell)}(\boldsymbol{r} \mid \boldsymbol{r}_0) \cdot V_p \quad (60)$$

The use of the pressure transfer function to express the radiated sound field becomes very useful when analyzing a system with several piston sources vibrating with different velocities. The following section presents an application example where this formulation is used.

VI. Application Example: Rigid Liner Splices

The mathematical model for sound radiation developed in this paper can be used to investigate many problems of practical significance in turbofan-engine noise. These potential applications are concerned with the discontinuities in liners such as due to scattering effects of liner splices, the effect of nonuniform wall impedance distributions, and so forth. Therefore, the considered discontinuity can be replaced by a sector of moving boundary that radiates sound into the lined circular duct with certain intensity. In fact, the difference among each application is reflected in the intensity of the sound radiated by such a device. In other words, every case will require a different way of computing the source velocity V_p needed by the model. The application example presented in this section consists of modeling the effect of a circumferential array of rigid patches on the surface of a uniform locally reacting liner.

A. Analytical Formulation

The disturbance incident noise field inside the duct consists of positive propagating modes expressed using Eq. (2) as

$$p_{\text{dist}}(\boldsymbol{r}) = \sum_{m=0}^{M_d} \sum_{n=0}^{N_d} (A_{mn}^{(+)})_{\text{dist}} \Phi_{mn}^{(+)}(r, \theta) e^{-ik_z^{(+)} z} e^{i\omega t} \quad (61)$$

The developed moving boundary formulation is used to model the effect of the rigid patches as piston sources adding to the present sound field. Because the problem is linear, the resulting sound field will be the superposition of the disturbance and piston sources. For example, for a single rigid patch (piston source) the sound field in the duct will be as

$$p(\boldsymbol{r}) = p_{\text{dist}}(\boldsymbol{r}) + p_{\text{piston}}^{(+)}(\boldsymbol{r} \mid \boldsymbol{r}_0) \quad (62a)$$

for the transmitted field (upstream) and

$$p(\boldsymbol{r}) = p_{\text{dist}}(\boldsymbol{r}) + p_{\text{piston}}^{(-)}(\boldsymbol{r} \mid \boldsymbol{r}_0) \quad (62b)$$

for the reflected field (downstream). The first term in Eqs. (62a) and (62b) represents the disturbance field in a uniform liner while the second represents the modification to the disturbance sound field due to the presence of the rigid patch. The expression for the radiated pressure $p_{\text{piston}}^{(\ell)}(\boldsymbol{r} \mid \boldsymbol{r}_0)$ was given in Eq. (60). In this expression, the piston velocities V_p must be selected to simulate the correct rigid boundary condition. Therefore, the fictitious piston source will have the strength required to cancel the existing radial particle velocity on the surface of every patch. This implies satisfying the following condition on every patch in the array:

$$-(v_{\text{dist}} + v_{\text{piston}})|_{\text{piston face}} + V_p = 0 \quad (63)$$

where v_{dist} is the particle velocity at the piston face induced by the disturbance sound field, and v_{piston} is the particle velocity at the same location induced by the piston itself. Note the negative sign for the radial particle velocities is introduced to match the inward normal convention (see Fig. 5). From knowledge of the pressure field, the radial particle velocities at the piston face are found using the definition of normalized acoustic admittance of the liner as in Eq. (14), that is, $v_r = (\beta_\omega/\rho c)p$. Thus, the rigid boundary condition stated in Eq. (63) can be rewritten in terms of the pressure transfer function defined in Eq. (60) as follows:

$$-\frac{\beta_w}{\rho c} p_{\text{dist}}(\boldsymbol{r}_0) - \frac{\beta_w}{\rho c} Z(\boldsymbol{r}_0 \mid \boldsymbol{r}_0) \cdot V_p + V_p = 0 \quad (64)$$

Note that Eq. (64) is imposed only at the center point of the piston. However, the induced particle velocity over the patch is not uniform and thus it is better to force the "average" particle velocity over the patch surface to vanish. This implies that, because the pistons are excited by the incoming sound, the range of frequencies that can be analyzed is limited by the relative dimension of the wavelength with respect to the size of the piston. The limit can be specified as the wavelength being equal to 2 times the size of the piston. For frequencies approaching this limit, the formulation will slowly start to underpredict the results until they completely average down to zero for higher frequencies. As a consequence, and depending on the range of frequencies of interest, cases where relatively large patches are considered, such as structural strips, require a surface discretization into smaller pistons and solve the problem with increased degrees of freedom.

The case of an array of rigid patches is then of more practical interest to present as an application example. The development for a single patch is then extended for an array of rigid rectangular-shaped patches, for example, as the one shown in Fig. 7. Similar to the single piston, the condition to satisfy is that the average particle velocity at the surface of all the patches in the array must vanish. To implement the formulation, the function Z_{os} will be defined as the average particle velocity transfer function over an "observer" piston due to the motion of another "source" piston. These functions are obtained

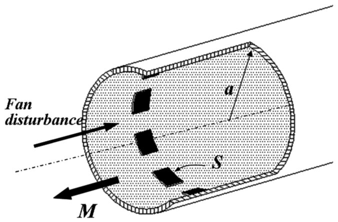

Fig. 7 Lined duct with a single circumferential array of rigid patches.

by integrating $Z(\boldsymbol{r} \mid \boldsymbol{r}_n)$ in Eq. (60) on the surface of the observer piston as follows:

$$Z_{\mathrm{os}} = \frac{1}{S_o} \int_{S_o} Z(\boldsymbol{r} \mid \boldsymbol{r}_s)\, \mathrm{d}S_o \tag{65}$$

where S_o is the observer piston area. To account for the interaction among all the rigid patches, the functions Z_{os} need to be arranged in a matrix form. Then, the zero average particle velocity condition over the patch surfaces [*equivalent to the condition in Eq. (64)*] is extended to the case of an array of patches by solving the following linear system of equations:

$$\{V_p\} - \frac{\beta_\omega}{\rho c}[Z_{\mathrm{os}}]_{B\times B} \cdot \{V_p\}_B = \frac{\beta_\omega}{\rho c}\{\bar{p}_{\mathrm{dist}}\}_B \tag{66}$$

where B is the number of patches. Upon the computation of the velocities V_p, the sound field is found from the superposition of the disturbance and piston source responses as expressed in Eqs. (62a) and (62b). In fact, both positive and negative propagating fields will be modified because the presence of the array of patches will always lead to reflection of acoustic energy back toward the fan and scatter of energy among both circumferential and radial propagating modes. To evaluate the performance of the liner system with the rigid patches, the acoustic power of the scattered modes can be computed. The expressions to compute the power can be found in the work by Alonso [24].

B. Numerical Results

This section illustrates the developed formulation with a numerical example consisting of an array of 10 rigid patches bonded to the surface of a lined duct. Note that if the number of patches is increased, it will approximate the case of a continuous hard-wall ring separating two liner sectors. The rigid patches in this example are square shaped with a surface $S = 16\ \mathrm{cm}^2$. The duct radius is assumed to be $a = 50$ cm and the circumferential array is axially located at 10 cm from the start of the liner. The liner total length is 36 cm and the liner core depth is $t = 1.08$ cm. The sound disturbance consists of only the mode order $m = 4$. In addition, there is a mean uniform flow of $M = -0.2$ inside the duct. The direction of the flow is assumed to be opposite to the positive direction of sound propagation, as shown in Fig. 7. The normalized liner impedance for the flow condition considered is given by the following expression:

$$Z_{\mathrm{liner}} = 1 + i\left[3.83E - 04 \cdot f - \cot\left(\frac{2\pi f}{c}t\right)\right] \tag{67}$$

The sound power propagation inside the described liner system is presented in Figs. 8–10. The results presented in each figure correspond to three different frequencies selected arbitrarily to illustrate the formulation, that is, 1545, 2045, and 2545 Hz, respectively. The incident acoustic power in the mode $m = 4$ scatters into a set of other modes at the location of the array. These scattered modes strictly follow the Tyler and Sofrin [28] relation as

$$m_{\mathrm{scatt.}} = m_{\mathrm{incident}} \pm k \cdot B \tag{68}$$

where k is any integer constant. Other modes do not receive any power from the incident mode. Also, note that the mode $m = 4$ is well cut on (in the hard-wall condition) for all cases and the reflected sound is not plotted. In addition, the scattered sound power decays

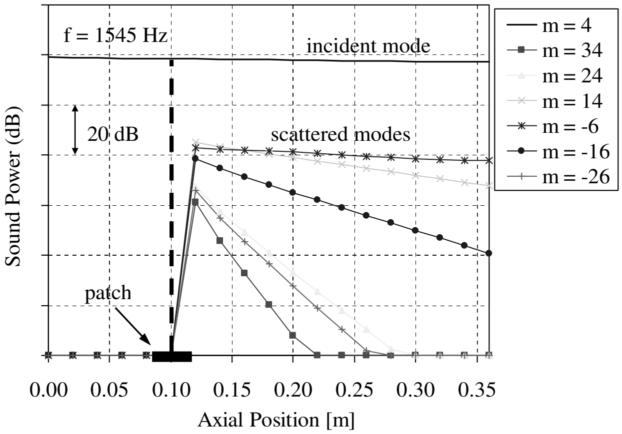

Fig. 8 Sound power propagation: scattering from an array of 10 rigid patches at 1545 Hz.

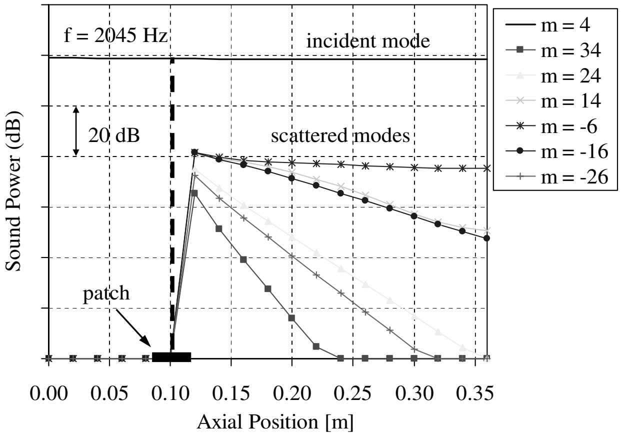

Fig. 9 Sound power propagation: scattering from an array of 10 rigid patches at 2045 Hz.

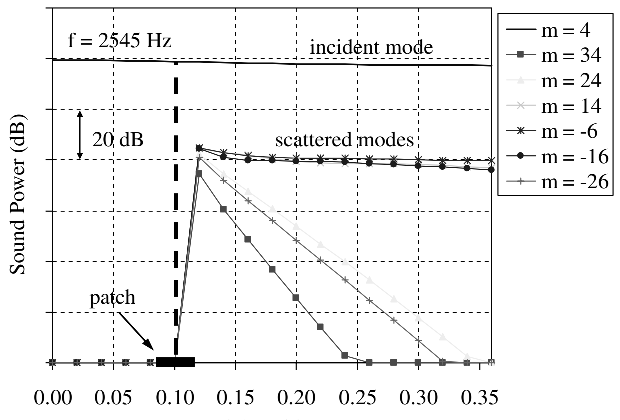

Fig. 10 Sound power propagation: scattering from an array of 10 rigid patches at 2545 Hz.

strongly for the higher order modes, which the liner is able to control effectively. Nevertheless, even when these modes are better attenuated, the fact must not be disregarded that the rigid splices reduce the effective area of acoustic treatment.

VII. Conclusions

A mathematical model to simulate the sound radiation due to a moving boundary inside a lined circular duct with uniform flow is developed. The formulation was derived by finding the Green's functions for a point source in a soft-wall duct with convective axial flow. To find the Green's functions in closed form, a modal expansion approach was used. The extension of the solution to finite piston radiation was performed using the divergence theorem. The adjoint solution to the nonhomogeneous wave equation was investigated in order to apply the reciprocity principle in the correct form. In fact, the direct solution does not satisfy reciprocity due to the nonsymmetry introduced by the flow. Therefore, the common Maxwell reciprocity must be applied using the adjoint solution.

The developed formulation was used to model the presence of an array of rigid patches bonded to the surface of the liner. If the shape of the piston sources is assumed to be simply rectangular, the problem can be solved in closed form. The analyzed example illustrated the scattering of acoustic power due to the discontinuities produced by the patches. In addition, the considered example is just one of the many applications of the presented modeling technique. The main difference in these applications lies in the way of computing the source velocity or strength required by the model. Some of the potential applications are in the modeling of actuators in active noise control in ducts, liner splices, liners with nonuniform impedance distribution, and so forth.

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